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CALDERÓN CONSTANTS OF FINITE-DIMENSIONAL COUPLES

BY

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ABSTRACT

The Calderón constant $æ(\bar{X})$ is a numerical invariant of finite-dimensional Banach couple $\bar{X} = (X_0, X_1)$ measuring its interpolation property with respect to linear operators acting in $\bar{X}$. In the paper we prove the duality relation $æ(\bar{X}) \approx æ(\bar{X}^*)$ and calculate the asymptotic behavior of $æ(\bar{X})$ as $\dim \bar{X} \to \infty$ for a few "classical" Banach couples.

1. Introduction

The aim of the paper is to study properties of a new invariant of a finite-dimensional Banach couple $\overline{X} = (X_0, X_1)$. For some reasons (see (1.3) below) it is named in the sequel the **Calderón constant** of $\overline{X}$ and is denoted by $æ(\overline{X})$. The origin of the notion and its applications lay in interpolation space theory but we will see that there are fruitful connections between this field of investigations and the local theory of Banach and metric spaces.

To introduce the basic concept consider the "doubled" Minkowski compact $\overline{\mathcal{M}}_n$ consisting of all n-dimensional Banach couples. An element $\overline{X}$ of $\overline{\mathcal{M}}_n$ is regarded as n-dimensional space X (over $\mathbb{R}$) equipped with two norms $\|\cdot\|_i$, so that $X_i = (X, \|\cdot\|_i)$, $i = 0, 1$.

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Let x, y be elements of X. We say that y is K-**majorized** by x in $\overline{X}$ (written $y \le x \bmod \overline{X}$ or simply $y \le x$ when no confusion can arise) if

$$K(t, y; \overline{X}) \le K(t, x; \overline{X}) \quad (t > 0). \tag{1.1}$$

See [BL] and [BK] for the definition of K-functional and other concepts and results of interpolation space theory; see also Section 2 below.

Definition 1.1: The Calderón constant of $\overline{X}$ is defined to be

$$æ(\overline{X}) := \sup_{y \le x} \inf_{T} \{\|T\|_{\overline{X}};\ y = Tx\}$$

where $\|T\|_{\overline{X}}$ stands for the norm of linear operator $T\colon X \to X$ in $\overline{X}$, i.e.

$$\|T\|_{\overline{X}} := \max_{i=0,1} \|T\|_{X_i \to X_i}.$$

The following natural questions related to the definition will be discussed in this paper:

(a) What is the attainable upper bound of $æ(\overline{X})$ on $\overline{\mathcal{M}}_n$?

It follows from Theorem 3.1 below that this quantity is equivalent to n as $n \to \infty$.

(b) What is the relation between $æ(\overline{X})$ and $æ(\overline{X}^*)$, where $\overline{X}^* := (X_0^*, X_1^*)$ is the dual couple?

Theorem 4.1 states that

$$æ(\overline{X}) \approx æ(\overline{X}^*)$$

with constants independent of $\dim X$.

(c) What is the asymptotic behavior of $æ(\overline{X})$ for the "classical" Banach couples?

The most important classical couple is $\left(\ell_{p_0}^n(w_0), \ell_{p_1}^n(w_1)\right)$ where

$$\|x\|_{\ell_p^n(w)} := \left\{\sum_{i=1}^{n} |x_i/w_i|^p\right\}^{1/p} \qquad (x \in \mathbb{R}^n).$$

But it was firstly proved by Calderón [C] that

$$æ(\ell_\infty^n, \ell_1^n) = 1 \tag{1.3}$$

and then the efforts of many mathematicians resulted in the Sparr theorem [Sp] which leads to the inequality

$$\text{æ}(\ell_{p_0}^n(w_0), \ell_{p_1}^n(w_1)) \leq 2. \tag{1.4}$$

It is worth pointing out that, by preceding results of Semenov and Sedaev [SS] and also of Sedaev alone [Se],

$$\text{æ}(\ell_p^n(w_0), \ell_p^n(w_1)) \leq 2^{1-1/p}. \tag{1.5}$$

Another important classical couple is the Lipschitz couple $(\ell_\infty(S), \mathrm{Lip}(S))$ where (S, d) is a finite metric space and

$$\|f\|_{\mathrm{Lip}(S)} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

Factorizing by constants one can consider this couple as an element of $\overline{\mathcal{M}}_{n-1}$ with $n = \operatorname{card} S$.

In Section 5 we will show that the asymptotic behavior of $\text{æ}(\overline{X})$ for Lipschitz couples is nontrivial and strongly connected with the metric properties of S. So, in the case of the metric subspace

$$\triangle_n := \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$$

of $\mathbb{R}$ we prove (Theorem 5.1) that

$$\text{æ}\,(\ell_\infty(\triangle_n), \mathrm{Lip}(\triangle_n)) \approx \log n \quad (n \to \infty).$$

We single out two corollaries of this result only.

Let $S = T_n$ be the dyadic tree of $n = 2^m - 1$ vertices with the metric induced by the tree structure (the distance of two adjacent vertices equals 1). Then we state (see Theorem 6.2) that

$$\text{æ}\,(\ell_\infty(T_n), \mathrm{Lip}(T_n)) \approx \log\log n \quad (n \to \infty).$$

Now let $\overline{W_p^n} := (\ell_p^n, v_p^n)$ be the discrete analog of the Sobolev couple $\big(L_p(0,1), W_p^1(0,1)\big)$, where

$$\|x\|_{v_p^n} := \left\{|x_1|^p + \sum_{k=1}^{n-1} |x_{k+1} - x_k|^p\right\}^{1/p} \quad (x \in \mathbb{R}^n).$$

Observing that $\mathrm{Lip}(\Delta_n) = v_\infty^n$ up to factorization by constants we obtain by duality

$$æ(\overline{W}_1^n) \approx \log n \quad (n \to \infty).$$

It is interesting to find the asymptotic of $æ(\overline{W}_p^n)$ for $1 < p < \infty$. Cwikel's result [Cw] suggests the following conjecture:

$$æ(\overline{W}_p^n) \approx (\log n)^{|1/2-1/p|} \quad (n \to \infty),$$

where $1 < p < \infty$, but now we can only prove a weaker result.

Finally, in Section 6 it will be proved that

$$æ\left(\ell_\infty(S), \mathrm{Lip}(S)\right) \leq 4\log(\#S)$$

for an arbitrary metric subspace of $\mathbb{R}$.

QUESTION: *Does a similar inequality hold for*

(a) *finite-pointed metric subspaces of* $\mathbb{R}^d$*?*

(b) *finite-pointed metric spaces?*

Some applications of the formulated results to the so-called "main problem" of the interpolation space theory are discussed in Section 7. The relevant definitions and results of this theory will be described in Section 2.

2. C-couples

Applications of the formulated results will be concerned with interpolation properties of Sobolev couples. For the convenience of the reader we present here the relevant material from the interpolation space theory. For details omitted we refer the reader to [BK] or [BL].

A pair $\overline{X} = (X_0, X_1)$ of Banach spaces is said to be a **Banach couple** if both X_i are continuously imbedded into a Hausdorff topological vector space.

A linear operator T: $X_0 + X_1 \to Y_0 + Y_1$ is said to map from $\overline{X}$ into $\overline{Y}$ if

$$T(X_i) \subset Y_i \quad (i = 0, 1)$$

and the norm

$$\|T\|_{\overline{X}\to\overline{Y}} := \max_{i=0,1} \{\|T|_{X_i}\|_{X_i \to Y_i}\}$$

is finite.

The notion of $\overline{X}$-majorization and Calderón constant $æ(\overline{X})$ introduced in Definition 1.1 for finite-dimensional couples can be readily extended to the general case. The norm $\|T\|_{\overline{X}}$ of this definition is clearly to be replaced by $\|T\|_{\overline{X}\to\overline{X}}$. Recall that the K-functional of $\overline{X}$ is defined by

$$K(t,x;\overline{X}) := \inf_{x=x_0+x_1} \{\|x_0\|_{X_0} + t\|x_1\|_{X_1}\} \quad (t>0)$$

where $x \in X_0 + X_1$.

Definition 2.1: $\overline{X}$ is called a C-**couple** if its Calderón constant is finite.

The next characterization of the set of all interpolation spaces of $\overline{X}$ is a direct consequence of the definitions. For its formulation recall that a Banach space X is called an **intermediate space** of $\overline{X}$ if

$$X_0 \cap X_1 \subset X \subset X_0 + X_1.$$

If in addition

$$\|T|_X\|_{X\to X} \leq \|T\|_{\overline{X}\to\overline{X}}$$

for every bounded linear operator $T: \overline{X} \to \overline{X}$, then X is said to be an **(exact) interpolation space** of $\overline{X}$.

Finally, the norm of X is called K_c-**monotone with respect to** $\overline{X}$ if the following property holds:

$$y \leq x \bmod \overline{X} \Longrightarrow \|y\|_X \leq c\|x\|_X.$$

PROPOSITION 2.2:

(a) *If the norm of X is K_1-monotone with respect to $\overline{X}$, then it is an exact interpolation space of $\overline{X}$.*

(b) *Conversely, if X is an exact interpolation space of $\overline{X}$, then its norm possesses K_c-property where $c = æ(\overline{X})$.*

Of course, the second statement is informative for C-couples only.

Calderón [C] was the first to characterize the set $\mathrm{Int}(L_1, L_\infty)$ of interpolation spaces of (L_1, L_∞) by the K-monotone property. Another description was proposed independently by Mitiagin [M]. Calderón's proof is based on the result equivalent to

$$æ(L_1, L_\infty) = 1.$$

Since then many other C-couples have been discovered; see a brief discussion below, and papers [CN] and [Ka] containing relevant references. For our goal the following constructive description of $\mathrm{Int}(\overline{X})$ proposed by Brudnyi and Kruglyak (see, e.g., [BK], Theorem 4.4.5) is of importance.

THEOREM 2.3: *Let $\overline{X}$ be a C-couple and let X be an exact interpolation space of $\overline{X}$. Then there is an absolute constant $c > 0$ and a Banach lattice ϕ of measurable functions on $(\mathbb{R}_+, dt/t)$ such that*

$$\|x\|_{K_\phi(\overline{X})} \leq \|x\|_X \leq c æ(\overline{X}) \|x\|_{K_\phi(\overline{x})}$$

for all $x \in X$.

Recall that the interpolation functor K_ϕ of the real method is defined by finiteness of the norm

$$\|x\|_{K_\phi(\overline{X})} := \|K(\cdot, x; \overline{X})\|_\phi.$$

Remark 2.4: The book [BK] contains a qualitative version of the theorem. But on checking the constants that appear in the proof we obtain the desired inequalities with $c < 23$.

Thus, in the case of C-couple $\overline{X}$ the set $\mathrm{Int}(\overline{X})$ coincides with the set $\{K_\phi(\overline{X})\}$ of all K-spaces of $\overline{X}$. Unfortunately, the problem of determining whether a given $\overline{X}$ is a C-couple, is very difficult. In accordance with our goal we present here a few results in this direction.

Let $L_p(w, d\mu)$ be a weighted L_p-space with the norm

$$\|f\| := \left\{ \int_\Omega \left| \frac{f}{w} \right|^p d\mu \right\}^{1/p}.$$

We denote this space by $L_p(w)$ if $(\Omega, d\mu) = (\mathbb{R}_+, dt/t)$ and we denote $L_p(w)$ by L_p^r if $w(t) := t^r$ $(t \in \mathbb{R}_+)$ where $0 \leq r \leq 1$.

Within this notation $\overline{X}_{w,p}$ and $\overline{X}_{rp}$ stand for $K_\phi(\overline{X})$ with $\phi = L_p(w)$ and L_p^r, respectively.

THEOREM 2.5 ([Sp]): *$\left(L_{p_0}(w_0), L_{p_1}(w_1)\right)$ is a C-couple.*

The result concluded a considerable amount of subsequent effort devoted to generalization of Calderón's result on weighted L_p-couples; see, e.g., [Ka] for the relevant references. The theorem was independently but a little later proved by Cwikel who applied it in order to prove the following striking result.

THEOREM 2.6 ([Cw]): *$(\overline{X}_{r_0p_0}, \overline{X}_{r_1p_1})$ is a C-couple for any $\overline{X}$ if $0 < r_0, r_1 < 1$.*

To formulate the generalization of this theorem we associate with the couple $(L_{p_0}(w_0), L_{p_1}(w_1))$ the pair $\{\omega_0, \omega_1\}$ of concave functions defined by

$$\omega_1(s) := \frac{1}{\|\min(1, t/s)\|_{L_{p_i}(w_i)}} \quad (s \in \mathbb{R}_+) \tag{2.2}$$

where $i = 0, 1$.

The pair $\{\omega_0, \omega_1\}$ of concave positive functions of $\mathbb{R}_+$ is said to be **factorable** if

$$\omega_i \approx \varphi_0^{1-r_i}\varphi_1^{r_i} \quad (i = 0, 1) \tag{2.3}$$

for some concave functions $\varphi_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ and numbers $r_i \in (0, 1)$ $(i = 0, 1)$.

Throughout the paper the equivalence of two functions $f \approx g$ means that

$$c^{-1}f \le g \le cf$$

for some $c > 0$. The infimum of such c is said to be the constant of the equivalence. Within the notion we may formulate

THEOREM 2.7:

(a) *If $\{\omega_0, \omega_1\}$ is factorable then $(\overline{X}_{w_0,p_0}, \overline{X}_{w_1,p_1})$ is a C-couple for any $\overline{X}$.*

(b) *The factorization condition is also necessary for $(\overline{X}_{w_0,p_0}, \overline{X}_{w_1,p_1})$ to be a C-couple in case $\overline{X} := (l_\infty(\mathbb{R}_+), \mathrm{Lip}(\mathbb{R}_+))$.*

The result has been proved in [BS].

Now let $\Lambda(\omega)$ be the Lipschitz space defined by

$$\|f\|_{\Lambda(\omega)} := \sup_{x \ne y} \frac{|f(y) - f(x)|}{\omega(|y - x|)}$$

where $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ is concave. We regard $\Lambda(\omega)$ as a Banach space by factorization by constants. In particular, $(\Lambda(\omega_0), \Lambda(\omega_1))$ with $\omega_i(t) := t^i$ $(i = 0, 1)$ coincides with the couple of the formulation of Theorem 2.7(b). Basing ourselves on the explicit form of K-functional for this couple, see [P], we deduce that

$$K_\phi(l_\infty(\mathbb{R}_+), \mathrm{Lip}(\mathbb{R}_+)) = \Lambda(\omega) \quad (\phi := L_\infty(\omega))$$

isometrically.

This leads to

COROLLARY 2.8: *$(\Lambda(\omega_0), \Lambda(\omega_1))$ is a C-couple iff (ω_0, ω_1) is factorable.*

The proof of Theorem 2.6(b) is based on the approach presented and may be treated as one of the applications of the method. In Section 7 we shall demonstrate some other applications restricting to the simplest case of Lipschitz couples of p-integrable functions $(\mathrm{Lip}_p(\mathbb{R}_+), L_p(\mathbb{R}_+))$ for $p = 1, \infty$. In case $1 < p < \infty$, i.e., when $\mathrm{Lip}_p(\mathbb{R}_+) = W_p^1(\mathbb{R}_+)$, the couple for the first time was studied by Cwikel [Cw], who proved that it was not a C-couple and, what is more, established a qualitative version of this statement. In case $p = \infty$ the same statement was proved by Bychkova [B] and by Cwikel and Mastylo [CM] in two different ways (see also Corollary 2.8 for the third proof of the result). But we shall also prove that the couple $(\mathrm{Lip}(\mathbb{R}_+), L_\infty(\mathbb{R}_+))$ is as bad as possible with respect to the C-property; see Theorem 7.1. The same fact is correct in case $p = 1$, but its proof requires a modification of the method presented. To avoid technical details we shall prove the weaker but also new result: $(\mathrm{Lip}_1(\mathbb{R}_+), L_1(\mathbb{R}_+))$ is not a C-couple, see Theorem 7.3.

3. Upper bound for n-dimensional Calderón constants

Within the notation of Section 1 let us denote

$$æ_n := \max\{æ(\overline{X}); \overline{X} \in \overline{\mathcal{M}}_n\}.$$

THEOREM 3.1: $n/2\sqrt{2} \le æ_n \le \sqrt{2}n$.

Proof (The upper estimate): We begin with the following general statement.

Let $\overline{X}$ be a **retract** of $\overline{Y}$, i.e. there are bounded linear operators $I: \overline{X} \to \overline{Y}$ and $P: \overline{Y} \to \overline{X}$ such that

$$PI = \mathrm{id}_{\overline{X}}. \tag{3.1}$$

LEMMA 3.2: $æ(\overline{X}) \le (\|I\| \cdot \|P\|)^2 æ(\overline{Y})$.

Proof: Suppose that $x, y \in X_0 + X_1$ satisfy

$$K(t, y; \overline{X}) \le K(t, x; \overline{X}) \quad (t > 0). \tag{3.2}$$

Then, it follows from (3.1) that

$$K(t, Iy; \overline{Y}) \le \|I\| K(t, y; \overline{X}) \le \|I\| K(t, PIx, \overline{X}) \le \|I\| \|P\| K(t, Ix; \overline{Y}).$$

From this and from Definition 1.1 of $æ(\overline{Y})$ it follows that $Iy = T(Ix)$ for a bounded linear operator $T := \overline{Y} \to \overline{Y}$ such that

$$\|T\| \le \|I\| \cdot \|P\|(1+\epsilon)æ(\overline{Y}) \quad (\epsilon > 0).$$

Thus, using (2.1) we conclude that

$$y = \hat{T}x := (PTI)x$$

and

$$\|\hat{T}\| \le \big(\|I\| \cdot \|P\|\big)^2(1+\epsilon)æ(\overline{Y}).$$

This gives the stated inequality. ∎

We are now in a position to show that for any n-dimensional couple $\overline{X}$ there exists a couple $\overline{H}$ of Hilbert spaces of the same dimension such that

$$æ(\overline{X}) \le n æ(\overline{H}). \tag{3.3}$$

Indeed, according to the John theorem, there exist euclidean norms $|\cdot|_i$ such that

$$|x|_i \le \|x\|_{X_i} \le \sqrt{n}|x|_i \quad (i = 0, 1). \tag{3.4}$$

Let $\overline{H} = (H_0, H_1)$ where $H_i = \big(X, |\cdot|_i\big)$ and let $I\colon \overline{X} \to \overline{H}$ and $P\colon \overline{H} \to \overline{X}$ be the identity maps. By (3.4), $\|P\| \cdot \|I\| \le \sqrt{n}$ and so, by applying Lemma 3.2, we get (3.3). Using now an appropriate basis in X we can reduce both the quadratic functionals $x \to |x|_i^2$ to a diagonal form. Thus, $\overline{H}$ is isometrically isomorphic to the couple $\big(l_2^n, l_2^n(w)\big)$ and by (1.5)

$$æ(\overline{H}) = æ\big(l_2^n, l_2^n(w)\big) \le \sqrt{2}. \tag{3.5}$$

This together with (3.3) leads to the desired upper bound.

The lower estimate: Fix $q > 1$ and define the space $l_{p,r}^n$ by the norm

$$\|(x_k)_1^n\|_{l_{p,r}^n} := \left\{ \sum_{k=1}^n |q^{-kr} x_k|^p \right\}^{1/p} \quad ((x_k)_1^n \in \mathbb{R}^n). \tag{3.6}$$

Given $n = 2m + e$ where $e \in \{0, 1\}$ we put

$$\overline{X} := \overline{l}_\infty^m \oplus \overline{l}_1^{m+e} \tag{3.7}$$

where one denotes

$$\bar{l}_p^s := \left(l_{p,0}^s, l_{p,1}^s\right).$$

We will establish that

$$æ(\overline{X}) \leq \frac{m+e}{\sqrt{2}} \leq \frac{n}{2\sqrt{2}}. \tag{3.8}$$

To prove the inequality in case $n = 2m$ (the remaining case is left to the reader) define $x, y \in \mathbb{R}^n$ by

$$x := (z, 0), \quad y := (0, z) \tag{3.9}$$

where

$$z := (1, \sqrt{q}, \ldots, (\sqrt{q})^m).$$

We begin with the inequality

$$K(t, y; \overline{X}) \leq \frac{\sqrt{q}+1}{\sqrt{q}-1} K(t, x; \overline{X}) \quad (t > 0). \tag{3.10}$$

It is well known and can be proved easily that

$$K\left(t, z; \bar{l}_1^m\right) = \sum_{k=1}^{m} q^{k/2} \min(1, q^{-k} t). \tag{3.11}$$

But the left side coincides with $K(t, y; \overline{X})$, see (3.9) and (3.7), and hence this function is a continuous broken line with knots q^i, $i = 1, 2, \ldots, m$, that equals 0 at 0 and is a constant for $t \geq q^m$. Since the right side of (3.10) is concave it suffices to check (3.10) at the points $t = q^i$, $i = 1, \ldots, m$.

From (3.11) we infer that

$$K(q^i, y; \overline{X}) = \sum_{k=1}^{i} q^{k/2} + q^{i/2} \sum_{k=i+1}^{m} q^{(i-k)/2} \leq \frac{\sqrt{q}+1}{\sqrt{q}-1} q^{i/2}.$$

On the other hand,

$$\begin{aligned} K(t, x; \overline{X}) = K(t, z; \bar{l}_\infty^m) &\geq \sup_{i \leq k \leq m} \left\{ \inf_{0 \leq \lambda \leq q^{k/2}} [\lambda + t(1-\lambda) q^{-k}] \right\} \\ &= \sup_{1 \leq k \leq m} \{ q^{k/2} \min(1, q^{-k} t) \}. \end{aligned}$$

The right side equals $q^{i/2}$ at $t = q^i$ as well. Therefore,

$$K(q^i, x; \overline{X}) \leq q^{i/2} \leq \frac{\sqrt{q}-1}{\sqrt{q}+1} K(q^i, y; \overline{X})$$

and (3.10) has been established.

From the inequality (3.10) it follows that

$$y = Tx$$

for some linear operator $T \colon \overline{X} \to \overline{X}$ satisfying

$$\|T\|_{\overline{X} \to \overline{X}} \leq \frac{\sqrt{q}+1}{\sqrt{q}-1} æ(\overline{X}).$$

Now let $P \colon \overline{X} \to \overline{l}_1^m$ and $I \colon \overline{l}_\infty^m \to \overline{X}$ be the canonical projection and injection respectively, i.e.

$$P(u \oplus v) = v, \quad I(u) = u \oplus 0.$$

Then their norms ≤ 1 and besides,

$$P(y) = z, \quad I(z) = x,$$

see (3.9). Hence the operator

$$S := PTI$$

maps from $\overline{l}_\infty^m$ into $\overline{l}_1^m$ and

$$\|S\|_{\overline{l}_\infty^m \to \overline{l}_1^m} \leq \frac{\sqrt{q}+1}{\sqrt{q}-1} æ(\overline{X}). \tag{3.12}$$

Besides, according to the definition of S

$$Sz = z. \tag{3.13}$$

Now we will use a special case of the Stein–Weiss interpolation theorem. It can also be proved directly by the classical Thorin trick; see, e.g., [BK], Section 1.7. In what follows we denote $\overline{X} \otimes \mathbb{C} := (X_0 \otimes \mathbb{C}, X_1 \otimes \mathbb{C})$ by $\overline{X}(\mathbb{C})$.

LEMMA 3.3: *Let T be a linear operator acting from $\bar{l}_\infty^m(\mathbb{C})$ into $\bar{l}_1^m(\mathbb{C})$ with the norm A. Then the norm of T as an operator from $l_{\infty,1/2}^m(\mathbb{C})$ into $l_{1,1/2}^m(\mathbb{C})$ is less than or equal to A.*

We apply the lemma to **complexification** S_c of the operator S defined by

$$S_c(x) := S(\operatorname{Re} x) + iS(\operatorname{Im} x), \quad x \in \mathbb{C}^m.$$

According to Krivine's result [Kr]

$$\|S_c\|_{\bar{l}_\infty^m(\mathbb{C}) \to \bar{l}_1^m(\mathbb{C})} \leq \sqrt{2}\|S\|_{\bar{l}_\infty^m \to \bar{l}_1^m}.$$

From this and (3.12) and (3.13) it can be concluded that

$$\|z\|_{l_{1,1/2}^m} \leq \sqrt{2}\frac{\sqrt{q}+1}{\sqrt{q}-1}æ(\overline{X})\|z\|_{l_{\infty,1/2}^m}.$$

But $z = (1, \sqrt{q}, \ldots, (\sqrt{q})^m)$ and hence its norm equals m on the left and 1 on the right. So, the inequality can be rewritten as follows:

$$æ_n \geq æ(\overline{X}) \geq \frac{\sqrt{q}-1}{\sqrt{q}+1}m = \frac{1}{2\sqrt{2}}\frac{\sqrt{q}-1}{\sqrt{q}+1}n.$$

This goes over to the stated lower estimate as $q \to \infty$. ∎

Remark 3.4: Is inequality (3.5) exact? It may be conjectured that

$$æ\big(l_2^n, l_2^n(w)\big) = 1.$$

It is worth pointing out the reformulation of the conjecture.

Let $x, y \in \mathbb{R}^n$ satisfy

$$\sum_{k=1}^n \frac{y_k^2 w_k^2}{1+t^2w_k^2} \leq \sum_{k=1}^n \frac{x_k^2 w_k^2}{1+t^2w_k^2} \quad (t > 0).$$

Does there exist a linear operator T mapping $\big(l_2^n, l_2^n(w)\big)$ into itself with the norm 1 such that $y = Tx$?

4. Duality

In what follows let $\overline{X} = (X_0, X_1)$ denote a fixed *finite-dimensional* Banach couple. If $X_i = (X, \|\cdot\|_i)$ we define the **dual Banach couple** $\overline{X}^* = (X_0^*, X_1^*)$ by putting $X_i^* := (X, \|\cdot\|_i^*)$ where $\|\cdot\|_i^* := \max\{\langle\cdot, y\rangle; \|y\|_i \leq 1\}$ is the dual Banach norm. It is important to point out that this definition coincides with the general one, see [BK], Section 2.4, restricted to the case of finite-dimensional couples. Therefore, we can and shall make use of the general duality theorems contained in [BK].

THEOREM 4.1: *There exists a constant $c > 0$ such that*

$$c^{-1}æ(\overline{X}) \leq æ(\overline{X}^*) \leq cæ(\overline{X}) \tag{4.1}$$

for any finite-dimensional Banach couple.

Proof: Since $\overline{X}^{**} = \overline{X}$ in this case, it suffices to prove the right inequality. The proof is based on the following propositions. To formulate the first of them, we recall the definition of **orbit** $\mathrm{Orb}_y(\overline{Y})$ of an element y in a Banach couple $\overline{Y}$, namely, the linear space

$$\mathrm{Orb}_y(\overline{Y}) := \{z \in Y_0 + Y_1; \quad \exists T\colon \overline{Y} \to \overline{Y}, z = Ty\}$$

equipped with the Banach norm

$$\|z\|_{\mathrm{Orb}_y(\overline{Y})} := \inf\{\|T\|_{\overline{Y}\to\overline{Y}}; z = Ty\}. \tag{4.2}$$

PROPOSITION 4.2: *Let x^* be an arbitrary non-zero element of $X_0^* + X_1^*$ $(= X)$. Then there exists an exact interpolation space A of $\overline{X}$ such that*

$$A^* = \mathrm{Orb}_{x^*}(\overline{X}^*)$$

isometrically.

The result is an immediate consequence of Theorem 2.3.34 of [BK]. We remark only that in this situation A^* coincides with $(X, \|\cdot\|_A^*)$.

On the other hand, according to Theorem 2.3 we get for this A the following:

PROPOSITION 4.3: *There exist a Banach lattice ϕ on $(\mathbb{R}_+, dt/t)$ and absolute constants, i.e., constants independent of $\overline{X}$, $c_1, c_2 > 0$ such that*

$$\begin{aligned} \|x\|_{K_\phi(\overline{X})} &\leq c_1 æ(\overline{X})\|x\|_A, \\ \|x\|_A &\leq c_2\|x\|_{K_\phi(\overline{X})}, \end{aligned} \tag{4.3}$$

for all $x \in X_0 + X_1$.

Now let the Banach lattice Ψ be defined by

$$\Psi := K_\phi(\overline{L}_1) \tag{4.4}$$

where ϕ is the Banach lattice of Proposition 4.3. Recall that the Banach couple $\overline{L}_p$ is equal to (L_p^0, L_p^1) where the space L_p^θ $(0 \le \theta \le 1)$ is defined by the norm $\{\int_{\mathbb{R}_+} |t^{-\theta} f(t)|^p \frac{dt}{t}\}^{1/p}$.

Introduce the associate lattice Ψ^+ of Ψ by

$$\|f\|_{\Psi^+} := \sup \left\{ \int_{\mathbb{R}_+} f(t) g\left(\frac{1}{t}\right) \frac{dt}{t}; \|g\|_\Psi \le 1 \right\}. \tag{4.5}$$

PROPOSITION 4.4: *There exist absolute constants* $c_3, c_4 > 0$ *such that*

$$\begin{aligned} \|z^*\|_{K_{\Psi^+}(\overline{X}^*)} &\le c_3 \|z^*\|_{\mathrm{Orb}_{x^*}(\overline{X})}, \\ \|z^*\|_{\mathrm{Orb}_{x^*}(\overline{X}^*)} &\le c_4 æ(\overline{X}) \|z^*\|_{K_{\Psi^+}(\overline{X}^*)}, \end{aligned} \tag{4.6}$$

for any $z^* = X_0^* + X_1^*$.

Here the orbit is taken from Proposition 4.2.

First of all we shall see how the proposition implies the theorem.

Assume elements $x^*, y^* \in X_0^* + X_1^*$ satisfy

$$K(t, y^*, \overline{X}^*) \le K(t, x^*; \overline{X}^*) \quad (t > 0).$$

Then, by the definition of a K-space, we see that

$$\|y^*\|_{K_{\Psi^+}(\overline{X}^*)} \le \|x^*\|_{K_{\Psi^+}(\overline{X})}.$$

Combining this with the first inequality (4.6) for $z^* := x^*$ and with the second one for $z^* := y^*$ we deduce that

$$\|y^*\|_{\mathrm{Orb}_{x^*}(\overline{X}^*)} \le c_4 æ(\overline{X}) \|y^*\|_{K_{\Psi^+}(\overline{X}^*)} \le c_3 c_4 æ(\overline{X}) \|x^*\|_{\mathrm{Orb}_{x^*}(\overline{X}^*)}.$$

But the norm on the right equals 1 and hence

$$\|y^*\|_{\mathrm{Orb}_{x^*}(\overline{X}^*)} \le c_3 c_4 æ(\overline{X}).$$

According to definition (4.2) there is a linear operator $T\colon \overline{X}^* \to \overline{X}^*$ such that

$$\|T\|_{\overline{X}^* \to \overline{X}^*} \le c_3 c_4 æ(\overline{X})$$

and $y^* = Tx^*$.

Remembering Definition 1.1 of the Calderón constant, we then can deduce that

$$æ(\overline{X}^*) \le c_3 x_4 æ(\overline{X}).$$

Thus, it remains to prove Proposition 4.4. To accomplish this we remark first of all that

$$J_\Psi(\overline{X}) \overset{1}{\hookrightarrow} K_\phi(\overline{X})$$

(see Theorem 3.43 of [BK]). The notion $X \overset{\gamma}{\hookrightarrow} Y$ means that the Banach space X imbeds into the Banach space Y with the imbedding constant less than or equal to γ. Passing to the dual spaces we get

$$K_\phi(\overline{X})^* \overset{1}{\hookrightarrow} J_\Psi(\overline{X})^*. \tag{4.7}$$

But, from the basic duality theorem of the real method (see [BK], Theorem 3.7.2),

$$J_\Psi(\overline{X})^* = K_{\Psi_+}(\overline{X}^*) \tag{4.8}$$

and the norms of these spaces are equivalent up to constant $c > 0$ independent of $\overline{X}$. Now it follows from (4.8), (4.7) and the second inequality (4.3) that

$$\|z^*\|_{K_{\Psi+}(\overline{X}^*)} \le c\|z^*\|_{J_\Psi(\overline{X})^*} \le c\|z^*\|_{K_\phi(\overline{X})^*} \le cc_2\|z^*\|_{A^*}$$

for every $z^* \in X_0^* + X_1^*$.

Together with the isometry of Proposition 4.2, this gives the first of the inequalities (4.6).

To prove the second inequality, we have to apply the imbedding

$$K_{\tilde\phi}(\overline{X}) \overset{18}{\hookrightarrow} J_{\tilde\Psi}(\overline{X}), \tag{4.9}$$

where $\tilde\Psi := K_{\tilde\phi}(\overline{L}_1)$ (see Theorem 3.5.5 and Remark 3.5.7 in [BK]). Unfortunately, the imbedding was proved for the so-called **non-degenerate** lattices $\tilde\phi$ only. This means the fulfilment of the condition:

$$\tilde\phi \setminus (L_\infty^0 \cup L_\infty^1) \ne \emptyset.$$

But this does not apply in our case because of the finiteness of the dimension of $\overline{X}$. To avoid the obstruction, we introduce the modified Banach lattice ϕ_λ by

$$\phi_\lambda := \phi + \lambda\tilde{\phi},$$

where $\tilde{\phi}$ is a fixed non-degenerate Banach lattice and the number $\lambda > 0$ will be chosen later on. Recall that the norm of this lattice is defined by

$$\|f\|_{\phi_\lambda} := \inf_{f=f_0+f_1} \{\|f_0\|_\phi + \lambda\|f_1\|_{\tilde{\phi}}\}. \tag{4.10}$$

It is clear ϕ_λ is non-degenerate and therefore

$$K_{\phi_\lambda}(\overline{X}) \overset{18}{\hookrightarrow} J_{\Psi_\lambda}(\overline{X}), \tag{4.11}$$

where

$$\Psi_\lambda := K_{\phi_\lambda}(\overline{L}_1). \tag{4.12}$$

According to the definition of ϕ_λ and Theorem 3.3.15 of [BK], the right side of (4.12) is equal to

$$K_\phi(\overline{L}_1) + \lambda K_{\tilde{\phi}}(\overline{L}_1) = \Psi + \lambda\tilde{\Psi},$$

where $\Psi := K_{\tilde{\phi}}(\overline{L}_1)$. In fact, this is an isometry because the constant of K-divisibility that is presented in the formulation of this theorem, is equal to 1 for the couple $\overline{L}_1$ (see [SS]).

Now using (4.8), (4.11) and the first inequality (4.3) we conclude that

$$\begin{aligned}\|z^*\|_{A^*} &\le c_2æ(\overline{X})\|z^*\|_{K_\phi(\overline{X})^*} \le c_2æ(\overline{X})\|z^*\|_{K_{\phi_\lambda}(\overline{X})^*} \\ &\le 18c_2æ(\overline{X})\|z^*\|_{J_{\Psi_\lambda}(\overline{X})^*} \le 18c_2æ(\overline{X})\|z^*\|_{K_{\Psi_\lambda}(\overline{X}^*)}.\end{aligned}$$

Here we denote

$$\Psi_\lambda := \Psi + \lambda\tilde{\Psi}$$

and take into account the imbedding

$$\phi \overset{1}{\hookrightarrow} \phi_\lambda.$$

According to definitions (4.5) and (4.10)

$$\Psi_\lambda^+ = \Psi^+ \cap \frac{1}{\lambda}\overline{\Psi}^+$$

with equivalence of norms (the equivalence constant ≤ 2). Recall that the right space defined by the norm

$$\|f\|_{\Psi^+\cap\frac{1}{\lambda}\tilde{\Psi}^+} := \max\{\|f\|_{\Psi^+}, \frac{1}{\lambda}\|f\|_{\tilde{\Psi}^+}\}.$$

Now, using the preceding inequality and the isometry of Proposition 4.2, we get

$$\|z^*\|_{\mathrm{Orb}_{x^*}(\overline{X}^*)} \leq 36c_2 \max\{\|z^*\|_{K_{\Psi^+}(\overline{X})}, \frac{1}{\lambda}\|z^*\|_{K_{\tilde{\Psi}^+}(\overline{X}^*)}\}.$$

Denote by S the unit sphere of $K_{\Psi^+}(\overline{X}^*)$. By the compactness of S

$$M := \sup_{z^*\in S} \|z^*\|_{K_{\tilde{\Psi}^+}(\overline{X}^*)} < \infty.$$

Putting $\lambda = 2M$ we deduce from the above inequality that

$$\|z^*\|_{\mathrm{Orb}_{x^*}(\overline{X}^*)} \leq 36c_2 æ(\overline{X}) \max\left\{1, \frac{1}{\lambda}M\right\} = c_4 æ(\overline{X})$$

for every $z^* \in S$.

By homogeneity of a norm, we thus conclude that

$$\|z^*\|_{\mathrm{Orb}_{x^*}(\overline{X}^*)} \leq c_4 æ(\overline{X})\|z^*\|_{K_{\Psi^+}(\overline{X}^*)}.$$

The proof of Proposition 4.4 is complete. ∎

To illustrate the theorem, we introduce the discrete analog (l_p^n, v_q^n) of the Sobolev couple (L_p, W_q^1). Recall that the second space is defined by

$$\|x\|_{v_q^n} := \{|x_1|^q + \sum_{k=1}^{n-1} |x_{k+1} - x_p|^q\}^{1/q}. \tag{4.13}$$

The couple (l_p^n, v_p^n) is denoted in Section 1 by $\overline{W}_p^n$ as well.

The dual norm to norm (4.13) is equal to

$$\|x\|_{(v_q^n)^*} = \left\{\sum_{k=1}^{n}\left|\sum_{s=1}^{k} x_s\right|^{q'}\right\}^{1/q'} \qquad (x \in \mathbb{R})$$

where $1/q + 1/q' = 1$.

Passing on to the new coordinates $y_s := \sum_{k=1}^{s} x_k$, we see that

$$\|x\|_{(v_q^n)^*} = \|y\|_{l_{q'}^n}, \quad \|x\|_{(l_p^n)^*} = \|y\|_{v_{p'}^n}.$$

Now, using Theorem 4.1, we deduce:

COROLLARY 4.5: *Uniformly in n, p, q*

$$æ(l_p^n, v_q^n) \approx æ(l_q^n, v_p^n).$$

In particular, within the notation of Section 1,

$$æ(l_\infty(\Delta_n), \mathrm{Lip}(\Delta_n)) \approx æ(\overline{W}_1^n). \tag{4.14}$$ ∎

5. Asymptotic for $æ(\overline{\Lambda}(\Lambda_n))$

Let M be a subset of $\mathbb{R}_+$ and $\overline{\Lambda}(M)$ denote the Lipschitz couple $(\Lambda_0(M), \Lambda_1(M))$. Recall that $\Lambda_\theta(M)$ is defined by the norm

$$\|f\|_{\Lambda_\theta(M)} := \sup_{t>0} \frac{\omega(t;f)}{t^\theta} \quad (0 \le \theta \le 1),$$

where

$$\omega(t;f) := \sup_{\substack{|a-b|\le t \\ a,b\in M}} |f(a) - f(b)|$$

is the modulus of continuity of f. We consider this as a Banach couple using factorization by constants.

Now let

$$\Delta_n := \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\right\}.$$

THEOREM 5.1: $æ(\overline{\Lambda}(\Delta_n)) \approx \log(n+1)$.

Proof: The upper estimate follows immediately from Theorem 6.1 (see also Proposition 6.3 for a slightly more exact result).

To prove the lower estimate we introduce the functions f and g_m by

$$f(t) := t^{1/2} \quad \text{and} \quad g_m(t) := c\sum_{i=1}^{m} 2^{-i/2} u_i(t) \quad (t \ge 0) \tag{5.1}$$

where $m := [\log_2 n] - 1$ and $c := (\sqrt{2}-1)2^{-5/2}$.

The sequence $\{u_i\}$ is defined by

$$u_i(t) := u(2^{i-1}t) \quad (i \in \mathbb{N}) \tag{5.2}$$

where $u\colon \mathbb{R}_+ \to \mathbb{R}$ is the broken line of period 1 such that

$$u(t) := |4t+1| \quad \text{if } -\tfrac{3}{4} < t \le \tfrac{1}{4}. \tag{5.3}$$

PROPOSITION 5.2:

(a) $g_m \in \Lambda_{1/2}(\mathbb{R}_+)$ *and, moreover,*

$$\omega(t; g_m) \leq \min\{2^{m/2}t, t^{1/2}, 1\}. \tag{5.4}$$

(b) *For every linear operator* $T: \overline{\Lambda}(\mathbb{R}_+) \to \Lambda(\mathbb{R}_+)$ *satisfying*

$$Tf = g_m$$

the inequality

$$\|T\|_{\overline{\Lambda}(\mathbb{R}_+)\to\overline{\Lambda}(\mathbb{R}_+)} \geq c_1 m$$

holds with an absolute constant $c_1 > 0$.

We first deduce the theorem from this proposition and then prove it.

Let $R_n: \overline{\Lambda}(\mathbb{R}_+) \to \overline{\Lambda}(\triangle_n)$ be the restriction operator and $E_n: \overline{\Lambda}(\triangle_n) \to \overline{\Lambda}(\mathbb{R}_+)$ the extension operator defined as follows:

(i) $E_n g$ is the linear function on $\left[\frac{j-1}{n}, \frac{j}{n}\right]$ that interpolates g at the ends ($j = 1, \ldots, n$);

(ii) $(E_n g)(t) := (E_n g)(1)$ $(t \geq 1)$.

Obviously E_n is a contraction, i.e., its norm ≤ 1.

Without loss of generality, we suppose hereafter that $n = 2^{m+1}$. Put $x := R_n f$ and $y := R_n g_m$. Note that by the choice of n,

$$E_n y = g_m.$$

Therefore, if a linear operator $S: \overline{\Lambda}(\triangle_n) \to \overline{\Lambda}(\triangle_n)$ takes x to y, then

$$(E_n S R_n) f = g_m.$$

Besides, according to assertion (b) of Proposition 5.2,

$$c_1 m \leq \|E_n S R_n\|_{\overline{\Lambda}(\mathbb{R}_+)\to\overline{\Lambda}(\mathbb{R}_+)} \leq \|S\|_{\overline{\Lambda}(\triangle_n)}.$$

If we now establish that

$$y \leq x \operatorname{mod}(\overline{\Lambda}(\triangle_n)) \tag{5.5}$$

then, by Definition 1.1 of the Calderón constant, we deduce that

$$æ(\overline{\Lambda}(\triangle_n)) \geq c_1 m \geq c_2 \log(n+1)$$

with some absolute constant $c_2 > 0$.

To prove (5.5) we make use of the identity

$$K(t, g: \overline{\Lambda}(M)) = \hat{\omega}(t; g) \quad (t \geq 0) \tag{5.6}$$

where $\hat{\omega}$ stands for the least concave majorant of ω (see [BK], Proposition 3.1.19). We mention that this proposition deals with the couple $(l_\infty(M), \mathrm{Lip}(M))$ but a simple change of the proof gives (5.6). So, according to (5.6) and (5.4),

$$K(t, y: \overline{\Lambda}(\triangle_n)) = \hat{\omega}(t, R_n g_m) \leq \hat{\omega}(t, g_m) \leq \min(t^{1/2}, 1) = \min(f(t), 1).$$

But $\hat{\omega}(t, R_n g_m)$ is a continuous concave broken line with knots j/n, $j = 1, \ldots, n$, equal to a constant, if $t \geq 1$. Therefore, the above inequality yields

$$K(t, y: \overline{\Lambda}(\triangle_n)) \leq (E_n R_n f)(t) \quad (t \geq 0).$$

But the right side is equal to $\omega(t, E_n R_n f)$ and

$$\omega(t, E_n R_n f) \leq \omega(t, R_n f) \leq K(t, R_n f; \overline{\Lambda}(\triangle_n)).$$

Remembering that $x := R_n f$ we see that this completes the proof of (5.5) and of the theorem.

Proof of Proposition 5.2: Since

$$\|u_i\|_{\Lambda_\theta(\mathbb{R}_+)} = 2 \cdot 2^{\theta i} \quad (\theta = 0, 1)$$

by the definition of u_i,

$$\omega(t, u_i) \leq \min_{\theta=0,1} \{t^\theta \|u_i\|_{\Lambda_\theta(\mathbb{R}_+)}\} = 2 \min_{\theta=0,1} (2^i t)^\theta$$

Hence,

$$\omega(t, g_m) \leq c \sum_{i=1}^m 2^{-i/2} \omega(t, u_i) \leq 2c \left\{ t \sum_{i=1}^l 2^{i/2} + \sum_{i=l+1}^m 2^{-i/2} \right\}$$
$$\leq 2c \left\{ \frac{2^{l/2} t}{1 - 1/\sqrt{2}} + \frac{2^{-(l+1)/2}}{1 - 1/\sqrt{2}} \right\}$$

for every l. Choose l satisfying $2^{-l-1} < t \leq 2^{-l}$. We then see from the above inequality that

$$\omega(t, g_m) \leq \frac{4\sqrt{2}}{\sqrt{2} - 1} c\sqrt{t} = \sqrt{t}$$

by the choice of c. Taking $l = 0$ or $l = m$ we complete the proof of part (a).

The proof of part (b) is based on two lemmas. The former is readily seen by direct calculation.

LEMMA 5.3: *$\int_0^1 u_i(t)u_j(t)dt = \delta_{ij}/3$, where δ_{ij} is the Kronecker symbol.*

To formulate the next result we introduce the sequence of functions $\{\varphi_i\}_{i\in\mathbb{Z}}$ by

$$\varphi_i(t) := \begin{cases} 2^{1-i}t - 1 & \text{if } 2^{i-1} \le t \le 2^i \\ 2^{-i/2}\sqrt{t} - \sqrt{2}(2^{-i}t - 1) & \text{if } 2^i < t \le 2^{i+1} \end{cases}$$

and $\varphi_i(t) := 0$ for other $t \in \mathbb{R}$.

Now let $T\colon \overline{\Lambda}(\mathbb{R}_+) \to \overline{\Lambda}(\mathbb{R}_+)$ be a bounded linear operator and let

$$\psi_i := T\varphi_i \quad (i \in \mathbb{Z}).$$

Then the following statement is valid.

LEMMA 5.4:

(a) *For $f(t) := t^{1/2}$ $(t \ge 0)$*

$$Tf = \sum_{i\in\mathbb{Z}} 2^{i/2}\psi_i \text{ (convergence in } \Lambda_0(\mathbb{R}_+) + \Lambda_1(\mathbb{R}_+)).$$

(b) *For every $t \ge 0$,*

$$\sum_{i\in\mathbb{Z}} |\psi_i(t)| \le 4\|T\|_{\Lambda_0(\mathbb{R}_+)}.$$

(c) $\sum_{i\in\mathbb{Z}} 2^i|\psi_i'(t)| \le \|T\|_{\Lambda_1(\mathbb{R}_+)}$ a.e. *on* $\mathbb{R}_+$.

Remark 5.5: In this formulation we regard elements of $\Lambda_i(\mathbb{R}_+)$ as functions which are equal to 0 at 0 $(i = 0, 1)$.

Proof: It is easy to verify that

$$0 \le \varphi_i \le 1, \quad \|\varphi_i\|_{\Lambda_1(\mathbb{R}_+)} = 2^{1-i} \quad (i \in \mathbb{Z})$$

and that

$$f = \sum_{i\in\mathbb{Z}} 2^{i/2}\varphi_i$$

where the series on the right converges in sum $\Lambda_0(\mathbb{R}_+) + \Lambda_1(\mathbb{R}_+)$. Since T is by definition a bounded linear operator in this sum, part (a) is established.

We remark now that supports of functions φ_i and φ_j are mutually disjoint if $|i-j| > 1$. Therefore, for every sequence $(\alpha_i)_{i\in\mathbb{Z}}$ with a finite support,

$$\Big\|\sum_{i\in\mathbb{Z}} \alpha_i\varphi_i\Big\|_{\Lambda_\theta(\mathbb{R}_+)} \le 2.2^{1-\theta} \sup_{i\in\mathbb{Z}} \|\alpha_i\varphi_i\|_{\Lambda_\theta(\mathbb{R}_+)} = 4 \sup_{i\in\mathbb{Z}} 2^{-i\theta}|\alpha_i|$$

where $\theta = 0, 1$.

According to Remark 5.5, $\psi_i := T\varphi_i$ equals 0 at 0. Therefore,

$$\Big|\sum_i \alpha_i \psi_i(t)\Big| \le \Big\|\sum_i \alpha_i \psi\Big\|_{\Lambda_0(\mathbb{R}_+)} \le 4\|T\|_{\Lambda_0(\mathbb{R}_+)} \sup_i |\alpha_i| \quad (t \ge 0).$$

Now fix $t \in \mathbb{R}_+$ and choose $\alpha_i := \operatorname{sign} \psi_i(t)$ if $|i| \le N$ and $\alpha_i := 0$ otherwise. Letting $N \to \infty$ we get assertion (b) from the above inequality.

Similarly, since

$$\|f\|_{\Lambda_1(\mathbb{R}_+)} = \|f'\|_{L_\infty(\mathbb{R}_+)}$$

we may conclude that

$$\Big|\sum_i \alpha_i \psi_i'(t)\Big| \le \Big\|\sum_i \alpha_i \psi\Big\|_{\Lambda_1(\mathbb{R}_+)} \le 4\|T\|_{\Lambda_1(\mathbb{R}_+)} \sup_i 2^{-i}|\alpha_i|,$$

for almost all $t \ge 0$.

Choosing in the inequality $\alpha_i := 2^i \operatorname{sign} \psi_i'(t)$ if $|i| \le N$ and $\alpha_i := 0$ otherwise and letting $N \to \infty$, we get assertion (c).

We now return to the proof of part (b) of Proposition 5.2. We attain this by estimating in two different ways the integral

$$\Omega := \int_0^1 g_m(t) \left(\sum_{j=1}^m 2^{j/2} u_j(t) \right) dt. \tag{5.7}$$

Using Lemma 5.3 we readily see that

$$\Omega = \tfrac{1}{3} cm.$$

Now we shall prove that

$$\Omega \le c_1 \|T\|_{\overline{\Lambda}(\mathbb{R}_+)} \tag{5.8}$$

where c_1 is an absolute constant. Comparing this with the preceding equality we obtain the desired estimate of $\|T\|_{\overline{\Lambda}(\mathbb{R}_+)}$.

So it remains to establish (5.8). To this end we insert the expression of g_m, i.e.,

$$g_m = \sum_{i \in \mathbb{Z}} 2^{i/2} \psi_i \quad (= Tf),$$

into integral (5.7) and divide the resulting sum into two parts:

$$\Omega = \sum_{k\in\mathbb{Z}} \sum_{j=1}^{m} 2^{k/2} \int_0^1 \psi_{k-j}(t) u_j(t) dt =: \Omega_0 + \Omega_1,$$

where $\Omega_0 := \sum_{k<0}$ and $\Omega_1 := \sum_{k\geq 1}$.

Applying inequality (b) of Lemma 5.4 and bearing in mind that $|u_k| \leq 1$, we see that

$$\Omega_0 \leq \sum_{k<0} 2^{k/2} \int_0^1 \Big(\sum_{j=1}^{m} |\psi_{k-j}(t)| \Big) \max_{1\leq j\leq m} |u_j| dt$$
$$\leq \frac{4}{\sqrt{2}-1} \|T\|_{\Lambda_0(\mathbb{R}_+)} \leq \frac{4}{\sqrt{2}-1} \|T\|_{\overline{\Lambda}(\mathbb{R}_+)}.$$

Now put

$$v_i(t) := \int_0^t u_i(s) ds.$$

According to the definition of u_i (see (5.2) and (5.3)),

$$0 \leq v_i \leq 2^{-i-1} \quad \text{and } v(0) = v(1) = 0.$$

We may now integrate by parts and apply inequality (c) of Lemma 5.4 to conclude that

$$\Omega_1 = \sum_{k\geq 0} 2^{k/2} \int_0^1 \sum_{j=1}^{m} \psi'_{k-j}(t) v_j(t) dt$$
$$\leq \sum_{k\geq 0} 2^{-k/2} \int_0^1 \Big(\sum_{j=1}^{m} 2^{k-j} |\psi'_{k-j}(t)| \Big) \max_{1\leq j\leq m} (2^j v_j(t) dt)$$
$$\leq \frac{2\sqrt{2}}{\sqrt{2}-1} \|T\|_{\Lambda_1(\mathbb{R}_+)} \leq \frac{2\sqrt{2}}{\sqrt{2}-1} \|T\|_{\overline{\Lambda}(\mathbb{R}_+)}.$$

This implies (5.6) and completes the proof of Proposition 5.2.

6. Extreme property of $\overline{\Lambda}(\triangle_n)$

In this section it will be shown that the couple $\overline{\Lambda}(\triangle_n)$ has asymptotically the largest Calderón constant among the couples $\overline{\Lambda}(M)$ defined on n-point subsets M of $\mathbb{R}$. More precisely, we prove

THEOREM 6.1: *For every n-point subset M of $\mathbb{R}$*

$$æ(\overline{\Lambda}(M)) \leq 4 \log n.$$

Proof: We prove by induction on $n := \#M$ the following more general

STATEMENT: Let $\overline{X} = (X_0, X_1)$ be an arbitrary Banach couple and let $x \in X_0 + X_1$ and $y: M \to \mathbb{R}$ satisfy

$$K(t, y; \Lambda(\overline{M})) \leq K(t, x; \overline{X}) \quad (t > 0).$$

Then there exists a linear operator $T: \overline{X} \to \overline{\Lambda}(M)$ taking x to y and satisfying

$$\|T\|_{\overline{X} \to \overline{\Lambda}(M)} \leq 4 \log(\#M).$$

The result is correct for $\#M = 2$. Indeed, $\dim \overline{\Lambda}(M) = 1$ in this case and we can consider $\overline{\Lambda}(M)$ as $(\mathbb{R}, \mathbb{R})$. Then the K-functional of y is equal to $|y| \min(1, t)$ and therefore $|y| \leq K(1, x; \overline{X})$. Applying the Hahn–Banach theorem, we may find a linear functional $f: X_0 + X_1 \to \mathbb{R}$ such that

$$|f(z)| \leq K(1, z; \overline{X})$$

for all z with the equality for $z = x$. If we then put

$$T(z) = \frac{f(z)}{K(1, x; \overline{X})} y,$$

we have obtained the desired operator T.

Suppose now that the statement is true for every $M \subset \mathbb{R}$ with $\#M = n - 1$ and we prove it for $M = \{a_1, \ldots, a_n\}$ $(n \geq 3)$. Here we assume that

$$a_1 < a_2 < \cdots < a_n.$$

To accomplish this we first reduce the proof to case $\overline{X} = \overline{L}_\infty$. To this end one remarks that $x \to K(\cdot, x; \overline{X})$ is a sublinear map from Banach space $X_0 + X_1$ into Banach lattice $L^0_\infty + L^1_\infty$. According to the Hahn–Banach–Kantorovich theorem, there exists a linear operator $U: X_0 + X_1 \to L^0_\infty + L^1_\infty$ such that

$$|Uz| \leq K(\cdot, z; \overline{X})$$

for all $z \in X_0 + X_1$ with the equality for $z = x$. Moreover,

$$\|Uz\|_{L^i_\infty} := \sup_{t>0} t^{-i}|(Uz)(t)| \le \sup_{t>0} \frac{K(t,x;\overline{X})}{t^i} \le \|z\|_i \quad (i = 0,1).$$

In other words, U maps $\overline{X}$ into $\overline{L}_\infty$ and its norm ≤ 1.

Therefore, in order to complete the proof, it suffices to find a linear operator S: $\overline{L}_\infty \to \overline{\Lambda}(M)$ such that

$$Sf = y$$

and, besides,

$$\|S\|_{\overline{L}_\infty \to \overline{\Lambda}(M)} \le 4\log n \tag{6.1}$$

and then put $T := SU$. In this situation, $f := K(\cdot, x; \overline{X})$ but we shall prove the assertion for any $f \in L^0_\infty + L^1_\infty$ satisfying

$$K(t, y, \overline{\Lambda}(M)) \le f(t) \quad (t \in \mathbb{R}_+). \tag{6.2}$$

For determining S we associate with each point $a_i \in M$ the restriction operator R_i: $\overline{\Lambda}(M) \to \overline{\Lambda}(M \setminus \{a_i\})$ and the extension operator E_i: $\overline{\Lambda}(M \setminus \{a_i\}) \to \overline{\Lambda}(M)$ given by

$$(E_i g)(a_i) := (1 - \alpha_i) g(a_{i-1}) + \alpha_i g(a_{i+1})$$

where

$$\alpha_i := \frac{a_i - a_{i-1}}{a_{i+1} - a_{i-1}} \quad (1 < i < n), \quad \alpha_1 = 1; \quad \alpha_n = 0.$$

Here we put $g(a_0) = g(a_{n+1}) := 0$.

It is readily seen that

$$\|E_i\|_{\overline{\Lambda}(M \setminus \{a_i\}) \to \overline{\Lambda}(M)} = 1 \quad (1 \le i \le n). \tag{6.3}$$

We also define the linear operator Λ_i: $\overline{L}_\infty \to \overline{\Lambda}(M)$ by

$$(\triangle_i g)(a_j) = [(1 - \alpha_i) g(a_i - a_{i-1}) + \alpha_i \varphi(a_{i+1} - a_i)]\delta_{ij},$$

where $1 \le j \le n$ (provided $g(a_1 - a_0) = g(a_{n+1} - a_n) := 0$).

It is easy to see that

$$\|\triangle_i\|_{L^1_\infty \to \Lambda_1(M)} \le 2\max\{\alpha_i, 1 - \alpha_i\}.$$

Since $0 \le \alpha_i \le 1$ and $\|\triangle_i\|_{L^0_\infty \to \Lambda_0(M)} \le 1$, we conclude that

$$\|\triangle_i\|_{\overline{L}_\infty \to \overline{\Lambda}(M)} \le 2 \quad (1 \le i \le n). \tag{6.4}$$

Identity (5.6) and inequality (6.3) imply

$$\begin{aligned}|y(a_i) - (E_i R_i y)(a_i)| &\le (1-\alpha_i)|y(a_i) - y(a_{i-1})| + \alpha_i |y(a_i) - y(a_{i+1})| \\ &\le (1-\alpha_i) f(a_i - a_{i-1}) + \alpha_i f(a_{i+1} - a_i).\end{aligned}$$

Hence we deduce that

$$y = E_i R_i y + \epsilon_i \triangle_i f \tag{6.5}$$

for some $\epsilon_i \in [-1,1]$ $(1 \le i \le n)$.

Since

$$K(t, f; \overline{L}_\infty) = \hat{f}(t) \ge f(t),$$

where $\hat{f}$ is the least concave majorant of f, inequality (6.2) yields

$$K(t, R_i y; \overline{\Lambda}(M \setminus \{a_i\})) \le K(t, f; \overline{L}_\infty) \quad (t \in \mathbb{R}_+).$$

According to the assumption of induction, there exists a linear operator $T_i \colon \overline{L}_\infty \to \overline{\Lambda}(M \setminus \{a_i\})$ such that

$$T_i f = R_i y \tag{6.6}$$

and, in addition,

$$\|T_i\|_{\overline{L}_\infty \to \overline{\Lambda}(M \setminus \{a_i\})} \le 4 \log(n-1). \tag{6.7}$$

We now define the required operator S by

$$S := \frac{1}{n} \sum_{i=1}^{n} (E_i T_i + \epsilon_i \triangle_i).$$

From (6.5) and (6.6) it follows that

$$Sf = y.$$

To estimate the norm of S one notes that support $\operatorname{supp}(\triangle_i \varphi)$ $(\varphi \in L^0_\infty + L^1_\infty)$ consists of one point at the most $(1 \le i \le n)$. Hence, in view of (6.4)

$$\left\| \sum_{i=1}^{n} \epsilon_i \triangle_i \right\|_{\overline{L}_\infty \to \overline{\Lambda}(M)} \le 2 \max_{1 \le i \le n} \|\triangle_i\|_{\overline{L}_\infty \to \overline{\Lambda}(M)} \le 4$$

and the inequality being combined with (6.3) and (6.7) gives

$$\|S\|_{\overline{L}_\infty \to \overline{\Lambda}(M)} \le 4\log(n-1) + \frac{4}{n} < 4\log n,$$

i.e., inequality (6.1) is proved. ∎

To demonstrate another asymptotic behavior of $æ(\overline{\Lambda}(M))$ we consider the dyadic tree T_n (where n is the number of vertices) with the metric d induced by the tree-structure of T_n. So $d(x,y)$ equals the number of edges of the shortest way in T_n connecting x and y.

THEOREM 6.2: $æ(l_\infty(T_n), \mathrm{Lip}(T_n)) \approx \log\log n$.

Proof: Let $b = \{v_1, \dots, v_m) \subset T_n$ be a branch of the maximal length. Let R be the restriction operator to b, i.e., $Rf := f|_b$, and E be the extension operator defined as follows. If $f\colon b \to \mathbb{R}$ and b_i is the branch emanating from $v_i \in b$, then we put

$$(Ef)(v) := f(v_i)$$

for every $v \in b_i$ $(1 \le i \le m-1)$. It is readily seen that R: $(l_\infty(T_n), \mathrm{Lip}(T_n)) \to (l_\infty(b), \mathrm{Lip}(b))$ and E acts in the inverse direction. Besides, norms of these operators ≤ 1. By Lemma 3.2 we therefore conclude that

$$æ(l_\infty(b), \mathrm{Lip}(b)) \le æ(l_\infty(T_n), \mathrm{Lip}(T_n)).$$

But b is isometrically isomorphic to the subspace $\{1, \dots, n\}$ of $\mathbb{R}$. Therefore the left side of the inequality is equal to $æ(\overline{\Lambda}(\triangle_m))$. Now, applying Theorem 5.1, we obtain the desired lower estimate

$$c_1 \log m \le æ(l_\infty(T_n), \mathrm{Lip}(T_n))$$

where $c_1 > 0$ is an absolute constant (and $n = 2^m$).

To accomplish the upper bound we need the following useful proposition, in which (S,d) is a finite-point metric space and

$$\delta(S) := \min_{x \ne y} d(x,y), \quad d(S) := \max_{x,y} d(x,y).$$

PROPOSITION 6.3: $æ(l_\infty(S), \mathrm{Lip}(S)) \le 3\log_2\left(\frac{2d(S)}{\delta(S)}\right)$.

Proof: It is clear that

$$\delta(S)\|f\|_{\mathrm{Lip}(S)} \le \|f\|_{l_\infty(S)} \le d(S)\|f\|_{\mathrm{Lip}(S)}$$

for any $f\colon M \to \mathbb{R}$. The norm of $l_\infty(S)$ is, of course, equal here to $\sup_{x,y\in S}|f(x) - f(x)|$.

Therefore, the statement follows from the next more general result.

Let $\overline{X}$ be a Banach couple such that

$$\alpha\|x\|_{X_0} \le \|x\|_{X_1} \le \beta\|x\|_{X_0} \quad (x \in X_0 + X_1) \tag{6.8}$$

where $\alpha, \beta > 0$ are constants.

LEMMA 6.4: $æ(\overline{X}) \le 3\log_2\left(\frac{2\beta}{\alpha}\right)$.

For the sake of completeness, we outline the (standard) proof of the result. Using (6.8) and the definition of K-functional, we conclude that

$$K(f, z; \overline{X}) = \begin{cases} t\|z\|_{X_1} & \text{if } t \le \alpha \\ \|z\|_{X_0} & \text{if } t \ge \beta \end{cases} \quad (z \in X_0 + X_1). \tag{6.9}$$

Now let x and y satisfy

$$K(f, y; \overline{X}) \le K(t, x; \overline{X}). \tag{6.10}$$

Denote the left side by $K(t)$ and suppose that $y = y_i^{(0)} + y_i^{(1)}$ is an optimal decomposition for $K(2^i\alpha)$, i.e.,

$$K(2^i\alpha) = \|y_i^{(0)}\|_{X_0} + 2^i\alpha\|y_i^{(1)}\|_{X_1}.$$

Here $i \in \{0, 1, \ldots, l\}$ and $l \in \mathbb{N}$ is defined by

$$2^{l-1}\alpha < \beta \le 2^l\alpha.$$

Then, from (6.9), it follows that

$$y_0^{(0)} = y_l^{(1)} = 0, \quad y_0^{(1)} = y_l^{(0)} = y.$$

Putting $y_i := y_i^{(0)} - y_{i-1}^{(0)} \left(= y_{i-1}^{(1)} - y_i^{(1)}\right)$ we get

$$y = \sum_{i=1}^{l} y_i$$

and

$$\|y_i\|_{X_s} \le \|y_i^{(s)}\|_{X_s} + \|y_{i-1}^{(s)}\|_{X_s} \le 3(2^i\alpha)^{-s}K(2^i\alpha, y; \overline{X}) \quad (s = 0, 1).$$

Bearing in mind (6.10) we now find a linear functional f_i: $X_0 + X_1 \to \mathbb{R}$ such that

$$f_i(x) = 1$$

and, besides,

$$|f_i(z)| \le \frac{K(2^i\alpha, z; \overline{X})}{K(2^i\alpha, x; \overline{X})} \quad (z \in X_0 + X_1).$$

If we now define T: $X_0 + X_1 \to X_0 + X_1$ by

$$Tz := \sum_{i=1}^{l} f_i(z) y_i,$$

then

$$T = \sum_{i=1}^{l} f_i(y) y_i = \sum_{i=1}^{l} y_i = y,$$

and moreover,

$$\|Tz\|_{X_s} \le 3\sum_{i=1}^{l}(2^i\alpha)^{-s}K(2^i\alpha, z; \overline{X}) \quad (s = 0, 1),$$

by the choice of f_i and (6.10). But each term on the right $\le \|z\|_{X_s}$ and hence

$$æ(\overline{X}) \le \|T\|_{\overline{X}\to\overline{X}} \le 3l \le 3\log_2\left(\frac{2\beta}{\alpha}\right). \quad ∎$$

Returning to the proof of Theorem 6.2, we remark that

$$\delta(T_n) = 1 \quad \text{and} \quad d(T_n) = 2m - 2 \le 2\log_2 n.$$

It remains to apply Proposition 6.2. ∎

Remark 6.5: Now let T be a finite tree and let the distance of adjusted vertices $x, y \in T$ equal $w(x, y) > 0$. The distance $d(x, y)$ of two arbitrary vertices x, y is measured by

$$d(x, y) = \inf_{\{x_i\}} \sum w(x_i, x_{i+1})$$

where the infimum is taken over all ways in T connecting x and y. In fact, we have proved that

$$æ(l_\infty(T), \mathrm{Lip}(T)) \approx \log d(T),$$

in case $w = 1$.

Of course this result is incorrect in general, but using a modification of the proof of Theorem 6.1, one can state that

$$æ(l_\infty(T), \mathrm{Lip}(T, d)) \le c \log(\#T)$$

with constant c independent of T and d.

7. Applications

As we have mentioned, $\overline{\Lambda}(\mathbb{R}_+)$ is not a C-couple (see [B] and [Cw]). Below we shall prove essentially a more exact result. For its formulation we have to use

Definition 7.1 [Cw]: $\overline{X}$ is said to be a C_p-**couple** $(1 \le p \le \infty)$ if for every $x, y \in X_0 + X_1$ satisfying

$$M_p(x,y) := \left\{ \int_{\mathbb{R}_+} \left[\frac{K(t, y; \overline{X})}{K(t, x; \overline{X})} \right]^p \frac{dt}{t} \right\}^{1/p} < \infty$$

there exists a linear operator $T\colon X_0 + X_1 \to X_0 + X_1$ such that

$$y = Tx,$$

and besides,

$$\|T\|_{\overline{X} \to \overline{X}} \le c M_p(x,y)$$

where c is an absolute constant.

It is clear that the notion of a C_∞-couple and a C-couple coincide. On the other hand, *every* Banach couple is a C_1-couple (see [Cw]). It is also proved in this paper that the couple $\overline{\Lambda}_p(\mathbb{R}_+) := (L_p(\mathbb{R}_+), \mathrm{Lip}_p(\mathbb{R}_+))$ is a C_p-couple with $q := 2p/|p-2|$ if $1 < p < \infty$. Here $\mathrm{Lip}_p(\mathbb{R}_+)$ is defined by the norm

$$\|f\|_{\mathrm{Lip}_p(\mathbb{R}_+)} := \sup_{t>0} \frac{\omega_p(f,t)}{t},$$

where

$$\omega_p(f,t) := \sup_{0<h<t} \left\{ \int_{\mathbb{R}_+} |f(x+h) - f(x)|^p dx \right\}^{1/p}$$

is modulus of continuity of f in L_p.

We remark that $q \to 2$ as $p \to \infty$. In contrast to this the following statement is valid.

THEOREM 7.2: *$\overline{\Lambda}(\mathbb{R}_+) := \overline{\Lambda}_\infty(\mathbb{R}_+)$ is not a C_p-couple for any $p > 1$.*

Proof: Suppose that $\overline{\Lambda}(\mathbb{R}_+)$ is a C_p-couple for a fixed $p > 1$. Then for functions $g_m(t)$ and $f(t) = t^{1/2}$ of Proposition 5.2 there exists a linear operator $T\colon \overline{\Lambda}(\mathbb{R}_+) \to \overline{\Lambda}(\mathbb{R}_+)$ such that

$$Tf = g_m, \tag{7.1}$$

and, besides,

$$\|T\|_{\overline{\Lambda}(\mathbb{R}_+)} \le cM_p(f, g_m), \tag{7.2}$$

where the constant c is absolute.

But by (5.6) and the choice of f we see that

$$K(t, f; \overline{\Lambda}(\mathbb{R}_+)) = \omega_\infty(f; t) = t^{1/2}.$$

In addition, by (5.6) and (5.4),

$$K(t, g_m; \overline{\Lambda}(\mathbb{R}_+)) \le \min\{2^{m/2}t, t^{1/2}, 1\}.$$

Hence, the right side of (7.2) is majorized by

$$c\left\{\int_{\mathbb{R}_+} \min\left\{2^{mp/2}t^{-1+p/2}, t^{-1}, t^{-1-p/2}\right\} dt\right\}^{1/p} \le c_1 m^{1/p}.$$

Therefore,

$$\|T\|_{\overline{\Lambda}(\mathbb{R}_+)} \le c_1 m^{1/p} \quad (p > 1)$$

contrary to (7.1) and assertion (b) of Proposition 5.2. ∎

A generalization of Proposition 5.2 can be used to prove a similar statement for the couple $\overline{\Lambda}_1(\mathbb{R}_+) = (L_1(\mathbb{R}_+), \mathrm{Lip}_1(\mathbb{R}_+))$. To avoid further technical details we confine ourselves to the following weaker result.

THEOREM 7.3: *$\overline{\Lambda}_1(\mathbb{R}_+)$ is not a C-couple.*

Proof: Let $B_n \subset L_1(\mathbb{R}_+)$ be the subspace of continuous piecewise affine functions with nodes at $0, 1, \ldots, n$, equal to 0 if $t \ge n$.

LEMMA 7.4: *There exists a linear projector* $\pi_n\colon L_1(\mathbb{R}_+) \to B_n$ $(n \in \mathbb{N})$ *such that*

$$\sup_n \|\pi_n\|_{\overline{\Lambda}_1} < \infty.$$

Proof: We choose a fixed continuous function φ on [0,1], which possesses the following property:

$$\int_0^1 \varphi(x)dx = 1, \quad \int_0^1 x\varphi(x)dx = 0. \tag{7.3}$$

The desired operator π_n is defined by

(a) $\pi_n f \in B_n$ $(f \in L_1(\mathbb{R}_+))$;

(b) $\pi_n(f;i) := \int_i^{i+1} \varphi(x-i)f(x)dx$, $i \in \{0,1,\dots,n-1\}$.

If $f \in B_n$, then $f(x) = A_i x + C_i$, $i \le x \le i+1$, and

$$\pi_n(f;i) = \int_i^{i+1} \varphi(x-i)(A_i x + C_i)dx = f(i) \quad (0 \le i \le n-1)$$

by (7.3). Besides, $\pi_n(f;n) = 0 = f(n)$ by the definition of B_n, and therefore,

$$\pi_n f = f \quad (f \in B_n).$$

We are now going to estimate the norm of π_n. First, by definition of π_n and the trapezoid rule,

$$\int_{\mathbb{R}_+} |\pi_n f| dx \le \sum_{i=0}^{n-1} |\pi_{2n}(f;i)| \le \max|\varphi| \int_0^n |f(x)|dx,$$

which gives

$$\|\pi_n\|_{L_1(\mathbb{R}_+)} \le \max|\varphi|.$$

The Lip_1-norm of $\pi_n f$ does not exceed

$$\int_{\mathbb{R}_+}^1 |\varphi(x)| \left(\sum_{i=0}^{n-1} |f(x+i+1) - f(x+i)| \right) dx \le \max|\varphi| \cdot \mathrm{Var}_{\mathbb{R}_+} f.$$

But the variation equals $\|f\|_{\mathrm{Lip}_1(\mathbb{R}_+)}$ by the Hardy–Littlewood theorem. Therefore,

$$\|\pi_n\|_{\mathrm{Lip}_1(\mathbb{R}_+)} \le \max|\varphi|. \qquad \blacksquare$$

Let us return to the proof of the theorem. Introduce the finite-dimensional couple $\overline{X}$ by

$$X_0 := (B_n, \|\cdot\|_{L_1}), \quad X_1 := (B_n, \|\cdot\|_{\mathrm{Lip}_1}).$$

According to Lemma 7.4, $\overline{X}$ is a retract of $\overline{\Lambda}_1$ and therefore Lemma 3.2 yields

$$æ(\overline{X}) \le (\max|\varphi|)^2 æ(\overline{\Lambda}_1(\mathbb{R}_+)).$$

Hence, by duality (see Theorem 4.1), we obtain

$$c_1 æ(\overline{X}^*) \le æ(\overline{\Lambda}_1(\mathbb{R}_+)), \tag{7.4}$$

the constant $c_1 > 0$ being independent of n. Simple calculation of the dual norms shows that

$$\left.\begin{aligned} \|f\|_{X_0^*} &= \sup_{0\le x\le n} |f(x)| \\ \|f\|_{X_1^*} &= \sup_{x\ge 0}\left|\int_0^x f(t)dt\right| \end{aligned}\right. \qquad (f \in B_n).$$

The differentiation $f \to f'$ maps this couple isometrically to the couple $\overline{Y} = (Y_0, Y_1)$ defined by

$$\|f\|_{Y_0} = \sup_{0\le i\le n-1} |f(i) - f(i+1)| = \|(f(i))\|_{\Lambda_1},$$
$$\|f\|_{Y_1} = \sup_{0\le x\le n} |f(x) - f(0)| = \sup_{1\le i\le n} |f(i) - f(0)| \approx \|(f(i))\|_{\Lambda_0}.$$

Therefore, Theorem 5.1 implies

$$c_1 \log n \le æ(\overline{Y}) = æ(\overline{X}^*).$$

Together with (7.4) this leads to the inequality

$$c_1 \log n \le æ(\overline{\Lambda}_1(\mathbb{R}_+))$$

where n is arbitrary, c_1 being independent of n. Letting $n \to \infty$, we complete the proof. ∎

References

[B] T. Bychkova, *Couples without C-properties*, in *Investigations in the Theory of Functions of Several Variables*, Yaroslavl, 1990, pp. 48–51 (in Russian).

[BK] Y. Brudnyi and N. Krugljak, *Interpolation Functors and Interpolation Spaces*, North-Holland, Amsterdam–New York–Oxford–Tokyo, 1992, 718 pp.

[BL] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Grundlehren der Mathematishe Wissenschaften **223**, Springer, Berlin–Heidelberg–New York, 1976, pp. 1–202.

[BS] Y. Brudnyi and A. Shteinberg, *Calderón couples of Lipschitz spaces*, Journal of Functional Analysis **131** (1995), 459–498.

[C] A. P. Calderón, *Spaces between L_1 and L_∞ and the theorems of Marcinkiewicz*, Studia Mathematica **26** (1966), 273–299.

[Cw] M. Cwikel, *Monotonicity properties of interpolation spaces II*, Arkiv für Matematik **19** (1981), 123–136.

[CM] M. Cwikel and M. Mastylo, *Interpolation spaces between the Lipschitz class and the space of continuous functions*, Proceedings of the American Mathematical Society **124** (1996), 1103–1110.

[CN] M. Cwikel and P. Nilsson, *Interpolation of weighted Banach lattices*, Memoirs of the American Mathematical Society (to appear).

[Ka] N. J. Kalton, *Calderón couples of rearrangement invariant spaces*, Studia Mathematica **106** (1993), 233–277.

[Kr] J.-L. Krivine, *Sur la complexification des opérateurs de L^∞ dans L^1*, Comptes Rendus de l'Académie des Sciences, Paris, Série A **284** (1977), 377–379.

[M] B. S. Mitiagin, *An interpolation theorem for modular spaces*, Matematicheskii Sbornik **66** (1965), 473–482 (in Russian).

[P] J. Peetre, *Exact interpolation theorems for Lipschitz continuous functions*, Ricerche di Matematica **18** (1969), 239–259.

[Se] A. Sedaev, *Description of interpolation spaces for the couple $(L^p_{\alpha_0}, L^p_{\alpha_1})$ and some related problems*, Doklady Akademii Nauk SSSR **209** (1973), 799–800 (in Russian).

[Sp] G. Sparr, *Interpolation of weighted L_p-spaces*, Studia Mathematica **62** (1978), 229–271.

[SS] A. A. Sedaev and E. M. Semenov, *On the possibility of describing interpolation spaces in terms of Peetre's K-method*, Optimizaciya **4** (1971), 98–114 (in Russian).